

REFINED AND MICROLOCAL KAKEYA-NIKODYM BOUNDS FOR EIGENFUNCTIONS IN TWO DIMENSIONS

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ABSTRACT. We obtain some improved essentially sharp Kakeya-Nikodym estimates for eigenfunctions in two-dimensions. We obtain these by proving stronger related microlocal estimates involving a natural decomposition of phase space that is adapted to the geodesic flow.

1. Introduction and main results.

Suppose that (M, g) is a two-dimensional compact Riemannian manifold and $\{e_\lambda\}$ are the associated eigenfunctions. That is, if Δ_g is the Laplace-Beltrami operator, we have

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x),$$

and, we assume throughout, that the eigenfunctions are normalized to have L^2 -norm one, i.e.,

$$\int_M |e_\lambda|^2 dV_g = 1,$$

where dV_g is the volume element.

The purpose of this paper is to obtain essentially sharp estimates that link, in two dimensions, the size of L^p -norms of eigenfunctions with $2 < p < 6$ to their L^2 -concentration near geodesics. Specifically, we have the following:

Theorem 1.1. *For every $0 < \varepsilon_0 \leq \frac{1}{2}$ we have*

$$(1.1) \quad \|e_\lambda\|_{L^4(M)} \lesssim_{\varepsilon_0} \lambda^{\frac{\varepsilon_0}{4}} \|e_\lambda\|_{L^2(M)}^{\frac{1}{2}} \times \|e_\lambda\|_{KN(\lambda, \varepsilon_0)}^{\frac{1}{2}}$$

if

$$(1.2) \quad \|e_\lambda\|_{KN(\lambda, \varepsilon_0)} = \left(\sup_{\gamma \in \Pi} \lambda^{\frac{1}{2} - \varepsilon_0} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2} + \varepsilon_0}}(\gamma)} |e_\lambda|^2 dV \right)^{\frac{1}{2}}$$

Equivalently, if $\varepsilon_0 > 0$ then there is a $C = C(\varepsilon_0, M)$ so that

$$(1.3) \quad \|e_\lambda\|_{L^4} \leq C \lambda^{\frac{1}{8}} \|e_\lambda\|_{L^2(M)}^{\frac{1}{2}} \times \left(\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2} + \varepsilon_0}}(\gamma)} |e_\lambda|^2 dV \right)^{\frac{1}{4}},$$

and therefore if $\int_M |e_\lambda|^2 dV = 1$, we have for any $\varepsilon > 0$, there is a $C = C(\varepsilon, M)$ so that

$$(1.4) \quad \|e_\lambda\|_{L^4(M)} \leq C \lambda^{\frac{1}{8} + \varepsilon} \sup_{\gamma \in \Pi} \|e_\lambda\|_{L^2(\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)})}^{\frac{1}{2}} \leq C \lambda^{\frac{1}{16} + \varepsilon} \sup_{\gamma \in \Pi} \|e_\lambda\|_{L^4(\mathcal{T}_{\lambda^{-\frac{1}{2}}(\gamma)})}^{\frac{1}{2}}.$$

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Here, Π denotes the space of unit-length geodesics in M and the last factor in (1.2) involves averages of $|e_\lambda|^2$ over $\lambda^{-\frac{1}{2}+\varepsilon_0}$ tubes about $\gamma \in \Pi$. Also, for simplicity, we are only stating things here and throughout for eigenfunctions, but the results easily extend to quasi-modes using results from [14].

Note that if $\varepsilon_0 = \frac{1}{2}$, then (1.1) is equivalent to the eigenfunction estimates from [9]

$$\|e_\lambda\|_{L^4(M)} \lesssim \lambda^{\frac{1}{8}} \|e_\lambda\|_{L^2(M)},$$

which are saturated by highest weight spherical harmonics on the standard two-sphere. We also remark that, up to the factor $\lambda^{\frac{\varepsilon_0}{4}}$, the estimate (1.1) is saturated by both the highest weight spherical harmonics and zonal functions on S^2 .

Bourgain [1] (with a slight loss) and the second author [11] proved this type of inequality where the first norm in the right is raised to the $\frac{3}{4}$ power and the second to the $\frac{1}{4}$ power. The inequalities in [11] were not formulated in this way but easily lead to this result. The approach in this paper made an inefficient use of the Cauchy-Schwarz inequality to handle the “easy” term (not the bilinear one), which led to this loss. The strategy for proving (1.1) will be to make an angular dyadic decomposition of a bilinear expression and pay close attention to the dependence of the bilinear estimates in terms of the angles, which we shall exploit using a multi-layered microlocal decomposition of phase space.

Before turning to the details of the proof, let us record a few simple corollaries of our main estimate.

If $\{a_{\lambda_{j_k}}\}_{k=0}^\infty$ is a sequence depending on a subsequence $\{\lambda_{j_k}\}$ of the eigenvalues of Δ_g , then we say that

$$a_\lambda = o_-(\lambda^\sigma)$$

if there is some $\varepsilon > 0$ and $C < \infty$ such that

$$|a_\lambda| \leq C(1 + \lambda)^{\sigma-\varepsilon}.$$

Then, using the above theorem we get the following:

Corollary 1.2. *The following are equivalent*

$$(1.5) \quad \|e_{\lambda_{j_k}}\|_{L^4(M)} = o_-(\lambda_{j_k}^{\frac{1}{8}})$$

$$(1.6) \quad \sup_{\gamma \in \Pi} \|e_{\lambda_{j_k}}\|_{L^4(\mathcal{T}_{\lambda_{j_k}^{-\frac{1}{2}}}(\gamma))} = o_-(\lambda_{j_k}^{\frac{1}{8}})$$

$$(1.7) \quad \sup_{\gamma \in \Pi} \|e_{\lambda_{j_k}}\|_{L^2(\mathcal{T}_{\lambda_{j_k}^{-\frac{1}{2}}}(\gamma))} = o_-(1).$$

Also, if either

$$(1.8) \quad \sup_{\gamma \in \Pi} \int_\gamma |e_\lambda|^2 ds = O(\lambda_{j_k}^\varepsilon), \quad \forall \varepsilon > 0,$$

or

$$(1.9) \quad \sup_{\gamma \in \Pi} \|e_{\lambda_{j_k}}\|_{L^2(\mathcal{T}_{\lambda_{j_k}^{-\frac{1}{2}}}(\gamma))} = O(\lambda_{j_k}^{-\frac{1}{4}+\varepsilon}), \quad \forall \varepsilon > 0.$$

then

$$(1.10) \quad \|e_{\lambda_{j_k}}\|_{L^4(M)} = O(\lambda_{j_k}^\varepsilon), \quad \forall \varepsilon > 0.$$

Here, ds denotes arclength measure on γ .

Clearly (1.5) implies (1.6). Also, (1.7) follows from (1.6) and Hölder's inequality. Since (1.1) shows that (1.7) implies (1.5). The last part of the corollary is also an easy consequence of the Theorem.

Note also that (1.4) says that if $e_{\lambda_{j_k}}$ is a sequence of eigenfunctions with

$$\|e_{\lambda_{j_k}}\|_{L^4(M)} = \Omega(\lambda_{j_k}^{\frac{1}{8}})$$

then for any ε there must be a sequence of shrinking geodesic tubes $\{\mathcal{T}_{\lambda_{j_k}^{-\frac{1}{2}}}(\gamma_k)\}$ for which for some $c = c_\varepsilon > 0$ we have

$$\|e_{\lambda_{j_k}}\|_{L^4(\mathcal{T}_{\lambda_{j_k}^{-\frac{1}{2}}}(\gamma_k))} \geq c \lambda_{j_k}^{\frac{1}{8}-\varepsilon}$$

In other words, up to a factor of $\lambda^{-\varepsilon}$ for any $\varepsilon > 0$, they fit the profile of the highest weight spherical harmonics by having maximal L^4 -mass on a sequence of shrinking $\lambda^{-\frac{1}{2}}$ tubes.

Like in Bourgain's estimate, (1.1) involves a slight loss, but this is not so important in view of the above application. In a later work we hope to show that (1.1) holds without this loss (in other words with $\varepsilon_0 = 0$), which should mainly involve refining the $S_{1/2,1/2}$ microlocal arguments that are to follow. Note that, because of the zonal functions on S^2 , this result would be sharp.

This paper is organized as follows. In the next section we shall introduce a microlocal Kekeya-Nikodym norm and an inequality involving it, (2.14), which implies (1.1). This norm is associated to a decomposition of phase space which is naturally associated to the geodesic flow on the cosphere bundle. In particular each term in the decomposition will involve bump functions which are supported in tubular neighborhoods of unit geodesics in S^*M . This decomposition and the resulting square function arguments are similar to the earlier ones in the joint paper of Mockenhaupt, Seeger and the second author [7], but there are some differences and new technical issues that must be overcome. We do this and prove our microlocal Kekeya-Nikodym estimate in §3. There after some pseudo-differential arguments we reduce matters to a oscillatory integral estimate which is a technical variation on the classical one in Hörmander [5], which was the main step in his proof of the Carleson-Sjölin theorem [3]. The result which we need does not directly follow from the results in [5]; however, we can prove it by adapting Hörmander's argument and using Gauss' lemma. After doing this, in §4 we shall see how our results are in some sense related to Zygmund's theorem [15] saying that in 2-dimensions eigenfunctions on the standard torus have bounded L^4 -norms. Specifically we shall see there that if we could obtain the endpoint version of (1.1), we would be able to recover Zygmund's theorem with no loss if we also knew a conjectured result that arcs on λS^1 of length $\lambda^{\frac{1}{2}}$ contain a uniformly bounded number of

In a later paper with S. Zelditch we hope to strengthen our results and also extend them to higher dimensions, as well as to present applications in the spirit of [13] of the

microlocal bounds which we obtain. The current authors would like to thank S. Zelditch for a number of stimulating discussions.

2. Microlocal Keakeya-Nikodym norms.

As in [11], [10, §5.1], we use the fact that we can use a reproducing operator to write $e_\lambda = \chi_\lambda f = \rho(\lambda - \sqrt{\Delta_g})e_\lambda$, for $\rho \in \mathcal{S}$ satisfying $\rho(0) = 1$, where, if $\text{supp } \hat{\rho} \subset (1, 2)$, we also have modulo $O(\lambda^{-N})$ errors (see [10, Lemma 5.1.3],

$$(2.1) \quad \chi_\lambda f(x) = \frac{1}{2\pi} \int \hat{\rho}(t) e^{i\lambda t} (e^{-it\sqrt{\Delta_g}} f)(x) dt = \lambda^{\frac{1}{2}} \int e^{i\lambda\psi(x,y)} a_\lambda(x,y) f(y) dV(y),$$

where

$$(2.2) \quad \psi(x,y) = d_g(x,y)$$

is the Riemannian distance function and if, as we may, we assume that the injectivity radius is 10 or more a_λ belongs to a bounded subset of C^∞ and satisfies

$$(2.3) \quad a_\lambda(x,y) = 0, \quad \text{if } d_g(x,y) \notin (1,2).$$

Thus, in order to prove (1.1), it suffices to work in a local coordinate patch and show that if a is smooth and satisfies the support assumptions in (2.3) and $0 < \delta < 1/10$ is small but fixed and if

$$x_0 = (0, y_0), \quad 1/2 < y_0 < 4,$$

is also fixed then

$$(2.4) \quad \left\| \lambda^{\frac{1}{2}} \int e^{i\lambda\psi(x,y)} a(x,y) f(y) dy \right\|_{L^4(B(0,\delta))}^2 \lesssim_{\varepsilon_0} \lambda^{\frac{\varepsilon_0}{2}} \|f\|_{L^2} \times \|f\|_{KN(\lambda, \varepsilon_0)}, \quad \text{if } \text{supp } f \subset B(x_0, \delta).$$

Here $B(x, \delta)$ denotes the δ -ball about x in our coordinates. We may assume that in our local coordinate system the line segment $(0, y)$, $|y| < 4$ is a geodesic.

In order to prove (2.4) we also need to define a microlocal version of the above Keakeya-Nikodym norm. We first choose $0 \leq \beta \in C_0^\infty(\mathbb{R}^2)$ satisfying

$$(2.5) \quad \sum_{\nu \in \mathbb{Z}^2} \beta(z + \nu) = 1, \quad \text{and } \text{supp } \beta \subset \{x \in \mathbb{R}^2 : |x| \leq 2\}.$$

To use this bump function, let $\Phi_t(x, \xi) = (x(t), \xi(t))$ denote the geodesic flow on the unit cotangent bundle. Then if (x, ξ) is a unit cotangent vector with $x \in B(x_0, \delta)$ and $|\xi_1| < \delta$, with δ small enough, it follows that there is a unique $0 < t < 10$ so that $x(t) = (s, 0)$ for some $s(x, \xi)$. If then for this t , $\xi(t) = (\xi_1(t), \xi_2(t))$, it follows that $\xi_2(t)$ is bounded from below. Let us then set $\varphi(x, \xi) = (s(x, \xi), \xi_1(t)/|\xi(t)|)$. Note that φ then is a smooth map from such unit cotangent vectors to \mathbb{R}^2 . Also, φ is constant on the orbit of Φ . Therefore, $|\varphi(x, \xi) - \varphi(y, \eta)|$ can be thought as measuring the distance from the geodesic in our coordinate patch through (x, ξ) to that of the one through (y, η) .

Let $\alpha(x)$ be a nonnegative C_0^∞ function which is one in $B(x_0, \frac{3}{2}\delta)$ and zero outside of $B(x_0, 2\delta)$. Given $\theta = 2^{-k}$ with $\lambda^{-\frac{1}{2}} \leq \theta \leq 1$, and $\nu \in \mathbb{Z}^2$ let $\Upsilon \in C^\infty(\mathbb{R})$ satisfy

$$(2.6) \quad \Upsilon(s) = 1, \quad s \in [c, c^{-1}], \quad \Upsilon(s) = 0, \quad s \notin [c/2, 2c^{-1}],$$

for some $c > 0$ to be specified later. We then put

$$(2.7) \quad Q_\theta^\nu(x, \xi) = \alpha(x) \beta(\theta^{-1}\varphi(x, \xi) + \nu) \Upsilon(|\xi|/\lambda).$$

This is a function of unit cotangent vectors, and we also denote its homogeneous of degree zero extension to the cotangent bundle with the zero section removed by $Q_\theta^\nu(x, \xi)$, $\xi \neq 0$, and the resulting pseudodifferential operator by $Q_\theta^\nu(x, D)$. Then if f is as in (2.4), we define its microlocal Keakeya-Nikodym norm corresponding to frequency λ and angle $\theta_0 = \lambda^{-\frac{1}{2}+\varepsilon_0}$ to be

$$(2.8) \quad \|f\|_{MKN(\lambda, \varepsilon_0)} = \sup_{\theta_0 \leq \theta \leq 1} \left(\sup_{\nu \in \mathbb{Z}^2} \theta^{-\frac{1}{2}} \|Q_\theta^\nu(x, D)f\|_{L^2(\mathbb{R}^2)} \right) + \|f\|_{L^2(\mathbb{R}^2)}, \quad \theta_0 = \lambda^{-\frac{1}{2}+\varepsilon_0}.$$

Note that

$$\sup_{\nu \in \mathbb{Z}^2} \theta^{-\frac{1}{2}} \|Q_\theta^\nu(x, D)f\|_{L^2(\mathbb{R}^2)}$$

measures the maximal microlocal concentration of f about all unit geodesics in the scale of θ . This is because of the fact that if we consider the restriction of Q_θ^ν to unit cotangent vectors and if $Q_\theta^\nu(x, \xi) \neq 0$, then $\text{supp } Q_\theta^\nu$ is contained in an $O(\theta)$ tube in the space of unit cotangent vectors about the orbit $t \rightarrow \Phi_t(x, \xi)$.

Let us collect a few facts about these pseudodifferential operators. First, the Q_θ^ν belong to a bounded subset of $S_{1/2+\varepsilon_0, 1/2-\varepsilon_0}^0$ (pseudodifferential operators of order zero and type $(1/2+\varepsilon_0, 1/2-\varepsilon_0)$), if $\lambda^{-\frac{1}{2}+\varepsilon_0} \leq \theta \leq 1$, with $\varepsilon_0 > 0$ fixed. Therefore, there is a uniform constant C_{ε_0} so that

$$(2.9) \quad \|Q_\theta^\nu(x, D)g\|_{L^2} \leq C_{\varepsilon_0} \|g\|_{L^2}, \quad \lambda^{-\frac{1}{2}+\varepsilon_0} \leq \theta \leq 1.$$

Similarly, if $P_\theta^\nu = (Q_\theta^\nu)^* \circ Q_\theta^\nu$, then by (2.5), for such θ , $\sum_\nu P_\theta^\nu$ belongs to a bounded subset of $S_{1/2+\varepsilon_0, 1/2-\varepsilon_0}^0$, and so we also have the uniform bounds

$$(2.10) \quad \left\| \sum_{\nu \in \mathbb{Z}^2} P_\theta^\nu(x, D)g \right\|_{L^2} \leq C_{\varepsilon_0} \|g\|_{L^2}, \quad \lambda^{-\frac{1}{2}+\varepsilon_0} \leq \theta \leq 1.$$

We can relate the microlocal Keakeya-Nikodym norm to the Keakeya-Nikodym norm if we realize that if the $\delta > 0$ above is small enough then there is a unit length geodesic γ_ν so that $Q_\theta^\nu(x, \xi) = 0$ for $x \notin \mathcal{T}_{C\theta_\nu}(\gamma)$, with C being a uniform constant. As a result, since $Q_\theta^\nu(x, \xi) = 0$ if $|\xi|$ is not comparable to λ , we can improve (2.9) and deduce that for every $N = 1, 2, \dots$, that there is a uniform constant C' so that we have

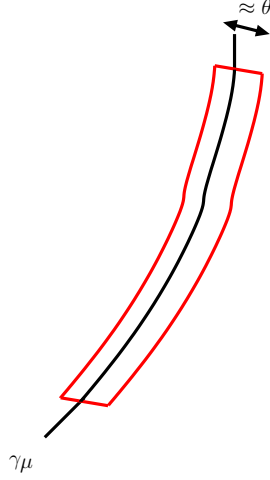
$$(2.11) \quad \|Q_\theta^\nu(x, D)g\|_{L^2} \leq C_{\varepsilon_0} \left(\int_{\mathcal{T}_{C'\theta}(\gamma_\nu)} |g|^2 dy \right)^{\frac{1}{2}} + C_N \lambda^{-N} \|g\|_{L^2}, \quad \lambda^{-\frac{1}{2}+\varepsilon_0} \leq \theta \leq 1,$$

since the kernel $K_\theta^\nu(x, y)$ of $Q_\theta^\nu(x, D)$ is $O(\lambda^{-N})$ for any N if y is not in $\mathcal{T}_{C'\theta}(\gamma_\nu)$, with C' sufficiently large but fixed. (See Figure 1.) Since

$$\theta^{-\frac{1}{2}} \left(\int_{\mathcal{T}_{C'\theta}(\gamma_\nu)} |g|^2 dy \right)^{\frac{1}{2}} \lesssim \sup_{\gamma \in \Pi} (\theta_0^{-1} \int_{\mathcal{T}_{\theta_0}(\gamma)} |g|^2 dy)^{\frac{1}{2}}, \quad \lambda^{-\frac{1}{2}+\varepsilon_0} = \theta_0 \leq \theta \leq 1,$$

we have

$$(2.12) \quad \sup_{\nu \in \mathbb{Z}^2} \theta^{-\frac{1}{2}} \|Q_\theta^\nu(x, D)f\|_{L^2(\mathbb{R}^2)} \leq C_{\varepsilon_0} \|f\|_{MKN(\lambda, \varepsilon_0)}, \quad \lambda^{-\frac{1}{2}+\varepsilon_0} \leq \theta \leq 1,$$

FIGURE 1. $\mathcal{T}_{C'\theta(\gamma_\nu)}$

meaning that we can dominate the microlocal Keakeya-Nikodym norm by the Keakeya-Nikodym norm.

From this, we conclude that we would have (2.4) if we could show

$$(2.13) \quad \left\| \int \lambda^{\frac{1}{2}} e^{i\lambda\psi(x,y)} a(x,y) f(y) dy \right\|_{L^4(B(0,\delta))}^2 \\ \lesssim_{\varepsilon_0} \lambda^{\frac{\varepsilon_0}{2}} \|f\|_{L^2} \times \|f\|_{MKN(\lambda,\varepsilon_0)}, \quad \text{if } \text{supp } f \subset B(x_0, \delta).$$

We note also that since $\chi_\lambda e_\lambda = e_\lambda$, this inequality of course yields the following microlocal strengthening of Theorem 1.1:

Theorem 2.1. *For every $0 < \varepsilon_0 \leq \frac{1}{2}$ we have*

$$(2.14) \quad \|e_\lambda\|_{L^4(M)} \lesssim_{\varepsilon_0} \lambda^{\frac{\varepsilon_0}{4}} \|e_\lambda\|_{L^2(M)}^{\frac{1}{2}} \times \|e_\lambda\|_{MKN(\lambda,\varepsilon_0)}^{\frac{1}{2}}.$$

if $\|e_\lambda\|_{MKN(\lambda,\varepsilon_0)}$ is as in (2.8).

3. Proof of the refined two-dimensional microlocal Keakeya-Nikodym estimates.

Let us now prove the estimates in (2.13). We shall follow arguments from §6 of [7].

We first note that if as in (2.4), $\text{supp } f \subset B(x_0, \delta)$, and if

$$(3.1) \quad \theta_0 = \lambda^{-\frac{1}{2} + \varepsilon_0}$$

with $\varepsilon_0 > 0$ fixed

$$\chi_\lambda f = \sum_{\nu \in \mathbb{Z}^2} \chi_\lambda (Q'_{\theta_0}(x, D) f) + R_\lambda f,$$

where, if $c > 0$ in (2.6) is small enough, and $N = 1, 2, 3, \dots$

$$\|R_\lambda f\|_{L^\infty} \lesssim \lambda^{-N} \|f\|_{L^2}.$$

Therefore, in order to prove (2.4), it suffices to show that

$$(3.2) \quad \left\| \sum_{\nu, \nu' \in \mathbb{Z}^2} \chi_\lambda Q_{\theta_0}^\nu f \chi_\lambda Q_{\theta_0}^{\nu'} f \right\|_{L^2} \lesssim_{\varepsilon_0} \lambda^{\frac{\varepsilon_0}{2}} \|f\|_{L^2} \times \|f\|_{MKN(\lambda, \varepsilon_0)}.$$

We shall split the sum in the left based on the size of $|\nu - \nu'|$. Indeed the left side of (3.2) is dominated by

$$(3.3) \quad \left\| \sum_{\nu} (\chi_\lambda Q_{\theta_0}^\nu f)^2 \right\|_{L^2} + \sum_{\ell=1}^{\infty} \left\| \sum_{|\nu - \nu'| \in [2^\ell, 2^{\ell+1})} \chi_\lambda Q_{\theta_0}^\nu f \chi_\lambda Q_{\theta_0}^{\nu'} f \right\|_{L^2}.$$

The square of the first term in (3.3) is

$$\sum_{\nu, \nu'} \int (\chi_\lambda Q_{\theta_0}^\nu f)^2 \overline{(\chi_\lambda Q_{\theta_0}^{\nu'} f)^2} dx.$$

Next we need an orthogonality result which is similar to Lemma 6.7 in [7], which says that if A is large enough we have

$$(3.4) \quad \sum_{|\nu - \nu'| \geq A} \left| \int (\chi_\lambda Q_{\theta_0}^\nu f)^2 \overline{(\chi_\lambda Q_{\theta_0}^{\nu'} f)^2} dx \right| \lesssim_{\varepsilon_0, N} \lambda^{-N} \|f\|_{L^2}^4.$$

We shall postpone the proof of this result until the end of the section when we will have recorded the information about the kernels of $\chi_\lambda Q_\theta^\nu$ that will be needed for the proof.

Since by [9]

$$\|\chi_\lambda\|_{L^2 \rightarrow L^4} = O(\lambda^{\frac{1}{8}}),$$

if we use (3.4) we conclude that the first term in (3.3) is majorized by (2.10) and (2.12) by

$$(3.5) \quad \begin{aligned} \lambda^{\frac{1}{2}} \sum_{\nu} \|Q_{\theta_0}^\nu f\|_{L^2}^2 \|Q_{\theta_0}^\nu f\|_{L^2}^2 + \lambda^{-N} \|f\|_{L^2}^4 &\lesssim \lambda^{\frac{1}{2}} \|f\|_{L^2}^2 \times \sup_{\nu \in \mathbb{Z}^2} \|Q_{\theta_0}^\nu f\|_{L^2}^2 + \lambda^{-N} \|f\|_{L^2}^4 \\ &= \lambda^{\varepsilon_0} \|f\|_{L^2}^2 \times \lambda^{\frac{1}{2} - \varepsilon_0} \sup_{\nu \in \mathbb{Z}^2} \|Q_{\theta_0}^\nu f\|_{L^2}^2 + \lambda^{-N} \|f\|_{L^2}^4 \end{aligned}$$

Therefore, the first term in (3.3) satisfied the desired bounds.

Using (2.12) again, the proof of (2.13) and hence (2.4) would be complete if we could estimate the other terms in (2.8) and show that for

$$(3.6) \quad \left\| \sum_{|\nu - \nu'| \in [2^\ell, 2^{\ell+1})} \chi_\lambda Q_{\theta_0}^\nu f \chi_\lambda Q_{\theta_0}^{\nu'} f \right\|_{L^2}^2 \lesssim_{\varepsilon_0} \|f\|_{L^2}^2 \times (2^\ell \theta_0)^{-1} \sup_{\nu \in \mathbb{Z}^2} \|Q_{2^\ell \theta_0}^\nu f\|_{L^2}^2 + \lambda^{-N} \|f\|_{L^2}^4.$$

Note that if $2^\ell \theta_0 \gg 1$ the left side of (3.6) vanishes and thus, as in (2.12), we are just considering $\ell \in \mathbb{N}$ satisfying $1 \leq 2^\ell \leq \lambda^{\frac{1}{2} - \varepsilon_0}$. In proving this, we may assume that ℓ is larger than a fixed constant, since the bound for small ℓ (with an extra factor of λ^{ε_0} in

the right) follows from what we just did. We can handle the sum over ℓ in (3.3) due to the fact that the right side of (3.6) does not include a factor λ^{ε_0} .

We now turn to estimating the non-diagonal terms in (3.3). We first note that by (2.5)

$$\chi_\lambda Q_{\theta_0}^\nu f = \sum_{\mu \in \mathbb{Z}^2} \chi_\lambda Q_\theta^\mu Q_{\theta_0}^\nu f + O_N(\lambda^{-N} \|f\|_2), \quad \text{if } \text{supp } f \subset B(x_0, \delta).$$

Furthermore, if, as we may, we assume that $\ell \in \mathbb{N}$ is sufficiently large, then given $N_0 \in \mathbb{N}$, there are fixed constants $c_0 > 0$ and $N_1 < \infty$ (with c_0 depending only on N_0 and the cutoff β in the definition of these pseudodifferential operators) so that if

$$\theta_\ell = \theta_0 2^\ell,$$

then

$$\begin{aligned} (3.7) \quad & \sum_{|\nu - \nu'| \in [2^\ell, 2^{\ell+1})} \chi_\lambda Q_{\theta_0}^\nu f \chi_\lambda Q_{\theta_0}^{\nu'} f \\ &= \sum_{\{\mu, \mu' \in \mathbb{Z}^2 : N_0 \leq |\mu - \mu'| \leq N_1\}} \sum_{|\nu - \nu'| \in [2^\ell, 2^{\ell+1})} \chi_\lambda Q_{c_0 \theta_\ell}^\mu Q_{\theta_0}^\nu f \chi_\lambda Q_{c_0 \theta_\ell}^{\mu'} Q_{\theta_0}^{\nu'} f + O_N(\lambda^{-N} \|f\|_{L^2}^2), \end{aligned}$$

for each $N \in \mathbb{N}$. Also, given $\mu \in \mathbb{Z}^2$, there is a $\nu_0(\mu) \in \mathbb{Z}^2$ so that

$$\|Q_{c_0 \theta_\ell}^\mu Q_{\theta_0}^\nu f\|_{L^2} \leq C_N \lambda^{-N} \|f\|_{L^2}, \quad \text{if } |\nu - \nu_0(\mu)| \geq C 2^\ell,$$

for some uniform constant C . If $|\mu - \mu'| \leq N_1$, then $|\nu_0(\mu) - \nu_0(\mu')| \leq C 2^\ell$ for some uniform constant C . Since $\|(Q_{\theta_0}^{\nu'})^* \circ Q_{\theta_0}^\nu\|_{L^2 \rightarrow L^2} = O(\lambda^{-N})$ for every N if $|\nu - \nu'|$ is larger than a fixed constant, it follows that

$$\begin{aligned} (3.8) \quad & \iint \left| \sum_{|\nu_0(\mu) - \nu|, |\nu_0(\mu') - \nu'| \leq C 2^\ell} \sum_{|\nu - \nu'| \in [2^\ell, 2^{\ell+1})} Q_{\theta_0}^\nu f(x) Q_{\theta_0}^{\nu'} f(y) \right|^2 dx dy \\ & \lesssim \sum_{|\nu - \nu_0(\mu)|, |\nu' - \nu_0(\mu')| \leq C' 2^\ell} \|Q_{\theta_0}^\nu f\|_{L^2}^2 \|Q_{\theta_0}^{\nu'} f\|_{L^2}^2 + O_N(\lambda^{-N} \|f\|_{L^2}^2), \quad \text{if } |\mu - \mu'| \leq C_0, \end{aligned}$$

for every N if C' is a sufficiently large but fixed constant. Also, using (2.10), we deduce that

$$\sum_{\mu \in \mathbb{Z}^2} \sum_{|\nu_0(\mu) - \nu| \leq C' 2^\ell} \|Q_{\theta_0}^\nu f\|_{L^2}^2 \lesssim \|f\|_{L^2}^2.$$

We clearly also have

$$\sum_{|\nu(\mu) - \nu'| \leq C' 2^\ell} \|Q_{\theta_0}^{\nu'} f\|_{L^2}^2 \lesssim \sup_{\mu \in \mathbb{Z}^2} \|Q_{2^\ell \theta}^\mu f\|_{L^2}^2.$$

Using these two inequalities and (3.8), we deduce that

$$\begin{aligned} (3.9) \quad & \sum_{|\mu - \mu'| \leq N_1} \left\| \sum_{|\nu_0(\mu) - \nu|, |\nu_0(\mu') - \nu'| < C 2^\ell} \sum_{|\nu - \nu'| \in [2^\ell, 2^{\ell+1})} Q_{\theta_0}^\nu f(x) Q_{\theta_0}^{\nu'} f(y) \right\|_{L^2(dx dy)} \\ & \lesssim \|f\|_{L^2} \times \sup_{\mu \in \mathbb{Z}^2} \|Q_{2^\ell \theta}^\mu f\|_{L^2} + O_N(\lambda^{-N} \|f\|_{L^2}^2). \end{aligned}$$

In addition to (3.4) we shall need another orthogonality result whose proof we postpone until the end of the section, which says that whenever θ is larger than a fixed positive multiple of θ_0 in (3.1) and N_1 is fixed

$$(3.10) \quad \left| \int (\chi_\lambda Q_\theta^\mu g_1 \chi_\lambda Q_\theta^{\mu'} g_2) \overline{(\chi_\lambda Q_\theta^{\tilde{\mu}} g_3 \chi_\lambda Q_\theta^{\tilde{\mu}'} g_4)} dx \right| \\ \lesssim_N \lambda^{-N} \prod_{j=1}^4 \|g_j\|_{L^2}, \quad \text{if } |\mu - \tilde{\mu}| + |\mu' - \tilde{\mu}'| \geq C, \text{ and } |\mu - \mu'|, |\tilde{\mu} - \tilde{\mu}'| \leq N_1,$$

for every $N = 1, 2, \dots$, with C being a sufficiently large uniform constant (depending on N_1 of course).

Using (3.9) and (3.10), we conclude that we would have (3.6) (and consequently (2.4)) if we could prove the following

Proposition 3.1. *Let*

$$(3.11) \quad (T_{\lambda, \theta}^{\mu, \mu'} F)(x) = \iint (\chi_\lambda Q_\theta^\mu)(x, y) (\chi_\lambda Q_\theta^{\mu'})(x, y') F(y, y') dy dy',$$

where

$$(\chi_\lambda Q_\theta^\mu)(x, y)$$

denotes the kernel of $\chi_\lambda Q_\theta^\mu$. Then if $\delta > 0$ is sufficiently small and if θ is larger than a fixed positive constant times θ_0 in (3.1) and if $N_0 \in \mathbb{N}$ is sufficiently large and if $N_1 > N_0$ is fixed, we have

$$(3.12) \quad \|T_{\lambda, \theta}^{\mu, \mu'} F\|_{L^2(B(0, \delta))} \lesssim_{\varepsilon_0} \theta^{-\frac{1}{2}} \|F\|_{L^2}, \quad \text{if } N_0 \leq |\mu - \mu'| \leq N_1, \\ \text{and } F(y, y') = 0, \text{ if } (y, y') \notin B(x_0, 2\delta) \times B(x_0, 2\delta).$$

To prove this we shall need some information about the kernel of $\chi_\lambda Q_\theta^\mu$. One thing will be that, by (2.7), the kernel is highly concentrated near the geodesic in M

$$(3.13) \quad \gamma_\mu = \{x_\mu(t) : -2 \leq t \leq 2, \Phi_t(x_\mu, \xi_\mu) = (x_\mu(t), \xi_\mu(t)), \theta^{-1} \varphi(x_\mu, \xi_\mu) + \mu = 0\},$$

which corresponds to Q_θ^μ . We also will exploit the oscillatory behavior of the kernel near γ_μ .

Specifically, we require the following

Lemma 3.2. *Let $\theta \in [C_0 \lambda^{-\frac{1}{2} + \varepsilon_0}, \frac{1}{2}]$, where C_0 is a sufficiently large fixed constant, and, as above, $\varepsilon_0 > 0$. Then there is a uniform constant C so that for each $N = 1, 2, 3, \dots$ we have*

$$(3.14) \quad |(\chi_\lambda Q_\theta^\mu)(x, y)| \leq C_N \lambda^{-N}, \quad \text{if } x \notin \mathcal{T}_{C\theta}(\gamma_\mu), \text{ or } y \notin \mathcal{T}_{C\theta}(\gamma_\mu).$$

Furthermore,

$$(3.15) \quad (\chi_\lambda Q_\theta^\mu)(x, y) = \lambda^{\frac{1}{2}} e^{i\lambda d_g(x, y)} a_{\mu, \theta}(x, y) + O_N(\lambda^{-N}),$$

where one has the uniform bounds

$$(3.16) \quad |\nabla_y^\alpha a_{\mu, \theta}(x, y)| \leq C_\alpha \theta^{-|\alpha|},$$

and

$$(3.17) \quad |\partial_t^j a_{\mu, \theta}(x, x_\mu(t))| \leq C_j, \quad x \in \gamma_\mu,$$

if, as in (3.13), $\{x_\mu(t)\} = \gamma_\mu$.

Proof. To prove the lemma it is convenient to choose Fermi normal coordinates so that the geodesic becomes the segment $\{(0, s) : |s| \leq 2\}$. Let us also write θ as

$$\theta = \lambda^{-\frac{1}{2}+\delta},$$

where, because of our assumptions $c_1 \leq \delta \leq 1/2$ for an appropriate $c_1 > 0$. Then, in these coordinates $Q_\theta^\mu(x, D)$ has symbol satisfying

$$(3.18) \quad q_\theta^\mu(x, \xi) = 0, \quad \text{if } |\xi_1/|\xi|| \geq C\lambda^{-\frac{1}{2}+\delta}, \quad |x_1| \geq C\lambda^{-\frac{1}{2}+\delta} \quad \text{or} \quad |\xi|/\lambda \notin [C^{-1}, C],$$

for some uniform constant C , and, additionally,

$$(3.19) \quad |\partial_{x_1}^j \partial_{x_2}^k \partial_{\xi_1}^l \partial_{\xi_2}^m q_\theta^\mu(x, \xi)| \leq C_{j,k,l,m} (1 + |\xi|)^{j(\frac{1}{2}-\delta)-l(\frac{1}{2}+\delta)-m}.$$

Next we recall that $\chi_\lambda = \rho(\lambda - \sqrt{-\Delta_g})$ where $\rho \in \mathcal{S}(\mathbb{R})$ satisfies $\hat{\rho} \subset (1, 2)$ and that the injectivity radius of (M, g) is ten or more. Therefore, we can use Fourier integral parametrices for the wave equation to see that the kernel of χ_λ is of the form

$$\chi_\lambda(x, y) = \iint e^{iS(t, x, \xi) - iy \cdot \xi + it\lambda} \hat{\rho}(t) \alpha(t, x, y, \xi) d\xi dt,$$

where $\alpha \in S_{1,0}^1$ and S is homogeneous of degree one in ξ is a generating function for the canonical relation for the half wave group $e^{-it\sqrt{-\Delta_g}}$. Thus,

$$(3.20) \quad \partial_t S(t, x, \xi) = -p(x, \nabla_x S(t, x, \xi)), \quad S(0, x, \xi) = x \cdot \xi,$$

and

$$(3.21) \quad \Phi_t(x, \nabla_x S) = (\nabla_\xi S, \xi),$$

with, as before, Φ_t denoting geodesic flow on the cotangent bundle. Furthermore,

$$(3.22) \quad \det \frac{\partial S}{\partial x \partial \xi} \neq 0.$$

By (3.18)-(3.19) and the proof of the Kohn-Nirenberg theorem, we have that

$$(3.23) \quad \begin{aligned} (\chi_\lambda Q_\theta^\mu)(x, y) &= \iint e^{iS(t, x, \xi) - iy \cdot \xi + i\lambda t} \hat{\rho}(t) q(t, x, y, \xi) d\xi dt + O(\lambda^{-N}), \\ &= \lambda^2 \iint e^{i\lambda(S(t, x, \xi) - y \cdot \xi + t)} \hat{\rho}(t) q(t, x, y, \lambda\xi) d\xi dt + O(\lambda^{-N}), \end{aligned}$$

where for all t in the support of $\hat{\rho}$,

$$(3.24) \quad q(t, x, y, \xi) = 0 \quad \text{if } |\xi_1/|\xi|| \geq C\lambda^{-\frac{1}{2}+\delta}, \quad |x_1| \geq C\lambda^{-\frac{1}{2}+\delta}, \quad \text{or } |\xi|/\lambda \notin [C^{-1}, C],$$

with C as in (3.19), and, also

$$(3.25) \quad |\partial_{x_1}^j \partial_{x_2}^k \partial_{\xi_1}^l \partial_{\xi_2}^m q(t, x, y, \xi)| \leq C_{j,k,l,m} (1 + |\xi|)^{j(\frac{1}{2}-\delta)-l(\frac{1}{2}+\delta)-m}.$$

Let us now prove (3.14). We have the assertion if $y \notin \mathcal{T}_{C\lambda^{-\frac{1}{2}+\delta}}(\gamma_\mu)$ by (3.24). To prove that remaining part of (3.24) which says that this is also the case when x is not in such a tube, we note that by (3.21), if $d_g(x_0, y_0) = t_0$ and $x_0, y_0 \in \gamma_\mu$, then

$$\nabla_\xi(S(t_0, x_0, \xi) - y_0 \cdot \xi) = 0 \quad \text{if } \xi_1 = 0.$$

By (3.22) we then have

$$|\nabla_\xi(S(t_0, x, \xi) - y_0 \cdot \xi)| \approx d_g(x, x_0) \quad \text{if } \xi_1 = 0.$$

We deduce from this that if $|\xi_1|/|\xi| \leq C\lambda^{-\frac{1}{2}+\delta}$, $|y_1| \leq C\lambda^{-\frac{1}{2}+\delta}$ and $|\xi| \in [C^{-1}, C]$, then there is a $c_0 > 0$ and a $C_0 < \infty$ so that

$$|\nabla_\xi(S(t_0, x, \xi) - y \cdot \xi)| \geq c_0\lambda^{-\frac{1}{2}+\delta} \quad \text{if } x \notin \mathcal{T}_{C_0\lambda^{-\frac{1}{2}+\delta}}(\gamma_\mu).$$

From this we obtain the remaining part of (3.14) via a simple integration by parts argument if we use the support properties (3.24) and size estimates (3.25) of $q(t, x, y, \xi)$. We note that every time we integrate by parts in ξ we gain by $\lambda^{-2\delta}$ which implies (3.14) since q vanishes unless $|\xi| \approx \lambda$ and δ is bounded below by a fixed positive constant.

To finish the proof of the lemma and obtain (3.15)-(3.17), we note that if we let

$$\Psi(t, x, y, \xi) = S(t, x, \xi) - y \cdot \xi + t$$

denote the phase function of the second oscillatory integral in (3.23), then at a stationary point where

$$\nabla_{\xi,t}\Psi = 0,$$

we must have $\Psi = d_g(x, y)$, due to the fact that $S(t, x, \xi) - y \cdot \xi = 0$ and $t = d_g(x, y)$ at points where the ξ -gradient vanishes. Additionally, it is not difficult to check that the mixed Hessian of the phase satisfies

$$\det \left(\frac{\partial^2 \Psi}{\partial(\xi, t) \partial(\xi, t)} \right) \neq 0$$

on the support of the integrand. This follows from the proof of [10, Lemma 5.1.3]. Moreover, since, modulo $O(\lambda^{-N})$ error terms $(\chi_\lambda Q_\theta^\mu)(x, y)$ equals

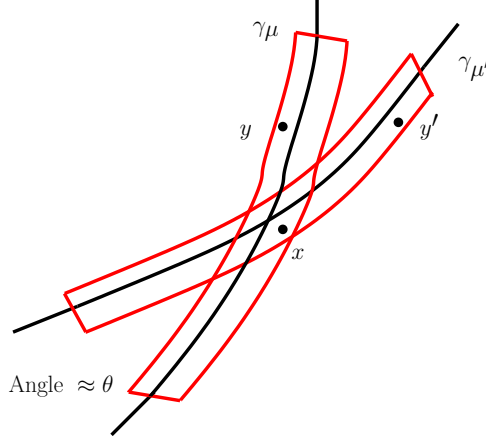
$$(3.26) \quad \lambda^2 \iint e^{i\lambda\Psi} \hat{\rho}(t) q(t, x, y, \lambda\xi) d\xi dt,$$

we obtain (3.15)-(3.16) by the proof of this result if we use stationary phase and (3.24)-(3.25). Indeed, by (3.21), (3.26) has a stationary phase expansion (see [6, Theorem 7.7.5]) where the leading term is a fixed constant times

$$(3.27) \quad \lambda^{\frac{1}{2}} e^{i\lambda t} q(t, x, y, \lambda\xi), \quad \text{if } t = d_g(x, y) \text{ and } \Phi_{-t}(y, \xi) = (x, \nabla_x S(t, x, \xi)).$$

From this, we see that the leading term in the asymptotic expansion must satisfy (3.16), and subsequent terms in the expansion will satisfy better estimates where the right hand side involves increasing negative powers of $\lambda^{2\delta}$ (by [6, (7.7.1)] and (3.25)), from which we deduce that (3.16) must be valid. Since $\xi_1 = 0$ and $p(y, \xi) = 1$ (by (3.21)) in (3.27) when $x, y \in \gamma_\mu$, we similarly deduce from (3.25) that the leading term in the stationary phase expansion must satisfy (3.17), and since the other terms satisfy better bounds involving increasing powers of $\lambda^{-2\delta}$, we similarly obtain (3.17), which completes the proof of the lemma. \square

Let us now collect some simple consequences of Lemma 3.2. First, in addition to (3.14), the kernel $(\chi_\lambda Q_\theta^\mu)(x, y)$ is also $O(\lambda^{-N})$ unless the distance between x and y is

FIGURE 2. θ -tubes intersecting at angle $\geq N_0\theta$

comparable to one by (2.3). From this we deduce that if $N_0 \in \mathbb{N}$ is sufficiently large

$$(3.28) \quad (\chi_\lambda Q_\theta^\mu)(x, y) (\chi_\lambda Q_\theta^{\mu'})(x, y') = O(\lambda^{-N}),$$

unless $\text{Angle}(x; y, y') \in [\theta, C_2\theta]$, and $x, y, y' \in \mathcal{T}_{C_2\theta}(\gamma_\mu)$, if $|\mu - \mu'| \in [N_0, N_1]$,

if $\text{Angle}(x, y, y')$ denotes the angle at x of the geodesic connecting x and y and the one connecting x and y' , and where $C_2 = C_2(N_1)$.

This is because in this case, if $x \in \mathcal{T}_{C\theta}(\gamma_\mu) \cap \mathcal{T}_{C\theta}(\gamma_{\mu'})$ then the tubes must be disjoint at a distance bounded below by a fixed positive multiple of θ if N_0 is large enough, and in this region their separation is bounded by a fixed constant times θ if N_1 is fixed. See the figure below.

To exploit this key fact, as above, let us choose Fermi normal coordinates about γ_μ so that the geodesic becomes the segment $\{(0, s) : |s| \leq 2\}$. Then, as in (2.2), let

$$\psi(x; y) = d_g((x_1, x_2), (y_1, y_2))$$

be the Riemannian distance function written in these coordinates. Then if x, y, y' are close to this segment and if the distance between x and y and x and y' are both comparable to one and if, as well, y is close to y' , it follows from Gauss' lemma that

$$(3.29) \quad \text{Angle}(x; (y_1, y_2), (y'_1, y'_2)) \approx \left| \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \psi(x, y) - \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \psi(x, y') \right|.$$

As a result, by (3.28), we have that there must be a constant $c_0 > 0$ so that

$$(3.30) \quad (\chi_\lambda Q_\theta^\mu)(x, y) (\chi_\lambda Q_\theta^{\mu'})(x, y') = O(\lambda^{-N}),$$

$$\text{if } \left| \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \psi(x, y) - \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \psi(x, y') \right| \leq c_0\theta, \text{ and } |\mu - \mu'| \in [N_0, N_1],$$

with, as above, $N_0 \in \mathbb{N}$ sufficiently large and N_1 fixed. Another consequence of Gauss' lemma is that if x and y , as in (3.29) are close to this segment and a distance which is

comparable to one from each other, then

$$(3.31) \quad \frac{\partial}{\partial x_1} \frac{\partial}{\partial y_1} \psi(x, y) \neq 0.$$

We shall also need to make use of the fact that, in these Fermi normal coordinates, we also have

$$(3.32) \quad \frac{\partial}{\partial x_2} \frac{\partial}{\partial y_1} \psi((0, x_2), (0, y_2)) = \frac{\partial}{\partial x_1} \psi((0, x_2), (0, y_2)) = 0, \\ \text{if } d_g((0, x_2), (0, y_2)) \approx 1.$$

Next, by (3.15)-(3.17), modulo terms which are $O(\lambda^{-N})$ we can write

$$(\chi_\lambda Q_\theta^\mu)(x, y) (\chi_\lambda Q_\theta^{\mu'}) (x, y') = \lambda e^{i\lambda(\psi(x, y) + \psi(x, y'))} b_\mu(x; y, y'),$$

where, by (3.28) and (3.30),

$$(3.33) \quad b_\mu(x; y, y') = 0, \text{ if } d_g(x, y) \text{ or } d_g(x, y') \notin [1, 2], \quad \text{or } |x_1| + |y_1| + |y'_1| \geq c_0^{-1}\theta, \\ \text{or } \left| \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \psi(x, y) - \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \psi(x, y') \right| \leq c_0\theta,$$

and, since we are working in Fermi normal coordinates

$$(3.34) \quad \left| \frac{\partial^j}{\partial x_1^j} \frac{\partial^k}{\partial x_2^k} b_\mu(x, y, y') \right| \leq C_0 \theta^{-j}, \quad 0 \leq j, k \leq 3.$$

The constants C_0 and c_0 can be chosen to be independent of $\mu \in \mathbb{Z}^2$ and $\theta \geq \lambda^{-\frac{1}{2} + \varepsilon_0}$ if $\varepsilon_0 > 0$. But then, by (3.33) and (3.34) if y_2 and y'_2 are fixed and close to one another, and if we set

$$\Psi(x; s, t) = \psi(x, (s+t, y_2)) + \psi(x, (s-t, y'_2)), \quad \text{and } b(x; s, t) = b_\mu(x; s+t, y_2, s-t, y'_2),$$

then we have that there is a fixed constant C so that

$$(3.35) \quad b(x; s, t) = 0, \quad \text{if } |x_1| + |s| + |t| \geq C\theta, \\ \text{and } \left| \frac{\partial^j}{\partial x_1^j} \frac{\partial^k}{\partial x_2^k} b(x; s, t) \right| \leq C\theta^{-j}, \quad 0 \leq j, k \leq 3,$$

while, by (3.31) and (3.32)

$$(3.36) \quad \frac{\partial}{\partial x_2} \frac{\partial}{\partial s} \Psi(0, x_2; 0, 0) = \frac{\partial}{\partial x_2} \frac{\partial}{\partial t} \Psi(0, x_2; 0, 0) = \frac{\partial}{\partial x_1} \Psi(0, x_2; 0, 0) = 0, \\ \text{but } \frac{\partial}{\partial x_1} \frac{\partial}{\partial s} \Psi(0, x_2; 0, 0) \neq 0, \quad \text{if } b(0, x_2; 0, 0) \neq 0,$$

and, moreover, by (3.33),

$$(3.37) \quad \left| \frac{\partial}{\partial x_2} \frac{\partial}{\partial t} \Psi(x; s, t) \right| \geq c\theta, \quad \text{if } b(x; s, t) \neq 0.$$

Also, if, as we may, because of the support assumption in (3.12), we assume that $|y_2 - y'_2| \leq \delta$ then

$$(3.38) \quad \left| \frac{\partial}{\partial x_1} \frac{\partial}{\partial t} \Psi(x; s, 0) \right| \leq C\delta, \quad \text{if } b(x; s, t) \neq 0,$$

since the quantity in the left vanishes identically when $y_2 = y'_2$.

Another consequence of Gauss' lemma is that if y, y', x are close to the 2nd coordinate axis and if the distance between both x and both y and y' are comparable to one then if θ above is bounded below the 2×2 mixed Hessian of the function $(x; y_1, y'_1) \rightarrow \psi(x, y) + \psi(x, y')$ has nonvanishing determinant. Thus, in this case (3.12) just follows from Hörmander's non-degenerate L^2 -oscillatory integral lemma in [5] (see [10, Theorem 2.1.1]). Therefore, it suffices to prove (3.12) when θ is bounded above by a fixed positive constant, and so Proposition 3.1 and hence Theorem 1.1 are a consequence of the following

Lemma 3.3. *Suppose that $b \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ vanishes when $|(s, t)| \geq \delta$. Then if $\Psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ is real and (3.35)–(3.38) are valid there is a uniform constant C so that if $\delta > 0$ and $\theta > 0$ are smaller than a fixed positive constant and*

$$T_\lambda F(x) = \iint e^{i\lambda\Psi(x;s,t)} b(x; s, t) F(s, t) ds dt,$$

then we have

$$(3.39) \quad \|T_\lambda F\|_{L^2(\mathbb{R}^2)} \leq C\lambda^{-1}\theta^{-\frac{1}{2}}\|F\|_{L^2(\mathbb{R}^2)}.$$

We shall include the proof of this result for the sake of completeness even though it is a standard result. It is a slight variant of the main lemma in Hörmander's proof of the Carleson-Sjölin theorem in [5] (see [10, pp. 61-62]). Hörmander's proof gives this result in the special case where $y_2 = y'_2$, and, as above, Ψ is defined by two copies of the Riemannian distance function. The case where y_2 and y'_2 are not equal to each other introduces some technicalities that, as we shall see, are straightforward to overcome.

Proof. Inequality (3.39) is equivalent to the statement that $\|T_\lambda^* T_\lambda\|_{L^2 \rightarrow L^2} \leq C\lambda^{-2}\theta^{-1}$. The kernel of $T_\lambda^* T_\lambda$ is

$$K(s, t; s', t') = \iint e^{i\lambda(\Psi(x;s,t) - \Psi(x;s',t'))} a(x; s, t, s', t') dx_1 dx_2,$$

$$\text{if } a(x; s, t, s', t') = b(x, s, t) \overline{b(x, s', t')},$$

Therefore, we would have this estimate if we could show that

$$(3.40) \quad |K(s, t; s', t')| \leq C\theta^{1-N} (1 + \lambda|(s - s', t - t')|)^{-N} \\ + C\theta(1 + \lambda\theta|(s - s', t - t')|)^{-N}, \quad N = 0, 1, 2, 3,$$

for then by using the $N = 0$ bounds for the regions where $|(s - s', t - t')| \leq (\lambda\theta)^{-1}$ and the $N = 3$ bounds in the complement, we see that

$$\sup_{s,t} \iint |K| ds' dt', \sup_{s',t'} \iint |K| ds dt \leq C\lambda^{-2}\theta^{-1},$$

which means that, by Young's inequality, $\|T_\lambda^* T_\lambda\|_{L^2 \rightarrow L^2} \leq C\lambda^{-2}\theta^{-1}$, as desired.

The bound for $N = 0$ follows from the first part of (3.35). To prove the bounds for $N = 1, 2, 3$, we need to integrate by parts.

Let us first handle the case where

$$(3.41) \quad |s - s'| \geq A^{-1}|t - t'|,$$

where $A \geq 1$ is a possibly fairly large constant which we shall specify in the next step. By the second part of (3.36) and by (3.38), we conclude that if $\delta > 0$ is sufficiently small (depending on A), we have

$$(3.42) \quad \left| \frac{\partial}{\partial x_1} (\Psi(x; s, t) - \Psi(x; s', t')) \right| \geq c|s - s'|, \quad |s - s'| \geq A^{-1}|t - t'|,$$

for some uniform constant $c > 0$.

Since $|K|$ is trivially bounded by the second term in the right side of (3.40) when $|s - s'| \leq (\lambda\theta)^{-1}$ and (3.41) is valid, we shall assume that $|s - s'| \geq (\lambda\theta)^{-1}$.

If we then write

$$(3.43) \quad e^{i\lambda(\Psi(x; s, t) - \Psi(x; s', t'))} = L e^{i\lambda(\Psi(x; s, t) - \Psi(x; s', t'))},$$

where $L(x, D) = \frac{1}{i\lambda(\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t'))} \frac{\partial}{\partial x_1},$

then we obtain

$$|K| \leq \iint |(L^*(x, D))^N a(x; s, t, s', t')| dx.$$

Note that

$$(3.44) \quad |\lambda(\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t'))|^N |(L^*)^N a|$$

$$\leq C_N \sum_{0 \leq j+k \leq N} \left| \frac{\partial^j}{\partial x_1^j} a \right| \times \sum_{\alpha_1 + \dots + \alpha_k \leq N} \frac{\prod_{m=1}^k \left| \frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}} (\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')) \right|}{|\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')|^k}.$$

Clearly,

$$(3.45) \quad \prod_{m=1}^k \left| \frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}} (\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')) \right| \leq C_k |(s - s', t - t')|^k,$$

and consequently, by (3.41) and (3.42),

$$(3.46) \quad \frac{\prod_{m=1}^k \left| \frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}} (\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')) \right|}{|\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')|^k} \leq C_{A,k}.$$

Since by (3.35), we have that $|\partial_{x_1}^j a| \leq C\theta^{-j}$, $j = 0, 1, 2, 3$, and (3.35) also says that a vanishes when $|x_1|$ is larger than a fixed multiple of θ , we conclude from (3.42)-(3.46) that if (3.41) holds then $|K|$ is dominated by the first term in the right side of (3.40).

We now turn to the remaining case which is

$$(3.47) \quad |t - t'| \geq A|s - s'|,$$

and where the parameter $A \geq 1$ will be specified. By the first part of (3.36) and by (3.37) and the fact that $|s|, |s'|, |t|, |t'|$ are bounded by a fixed multiple of θ in the support of a , it follows that we can fix A (independent of θ small) so that if (3.47) is valid then

$$\left| \frac{\partial}{\partial x_2} (\Psi(x; s, t) - \Psi(x; s', t')) \right| \geq c\theta|t - t'|, \quad \text{on supp } a,$$

for some uniform constant $c > 0$. Then since (3.32) implies that

$$\prod_{m=1}^k \left| \frac{\partial^{\alpha_m}}{\partial x_2^{\alpha_m}} (\Psi'_{x_2}(x; s, t) - \Psi'_{x_2}(x; s', t')) \right| \leq C_k \theta^k |(s - s', t - t')|^k, \quad \text{on supp } a,$$

and since, by (3.35),

$$|\partial_{x_2}^j a| \leq C_N, \quad 1 \leq j \leq N,$$

we conclude that, if we repeat the argument just given but now integrate by parts with respect to x_2 instead of x_1 , then $|K|$ is bounded by second term in the right side of (3.40), which completes the proof of Lemma 3.3. \square

To conclude matters, we also need to prove the orthogonality estimates (3.4) and (3.10). Since (3.4) is a special case of (3.10), we just need to establish the latter.

To see this, we note that by Lemma 3.2, if $(\chi_\lambda Q_\theta^\mu)(x, y)$ denotes the kernel of $\chi_\lambda Q_\theta^\mu$, then

$$\begin{aligned} (\chi_\lambda Q_\theta^\mu)(x, y) (\chi_\lambda Q_\theta^{\mu'})(x, y') \overline{(\chi_\lambda Q_\theta^{\tilde{\mu}})(x, \tilde{y}) (\chi_\lambda Q_\theta^{\tilde{\mu}'})(x, \tilde{y}')} &= O_N(\lambda^{-N}) \\ \text{if } x \notin \mathcal{T}_{C\theta}(\gamma_\mu) \cap \mathcal{T}_{C\theta}(\gamma_{\mu'}) \cap \mathcal{T}_{C\theta}(\gamma_{\tilde{\mu}}) \cap \mathcal{T}_{C\theta}(\gamma_{\tilde{\mu}'}), \end{aligned}$$

with C sufficiently large and the geodesics defined by (3.13). On the other hand, if x is in the above intersection of tubes, then the condition on $(\mu, \mu', \tilde{\mu}, \tilde{\mu}')$ in (3.10) ensures that if the constant C there is large enough we have

$$\begin{aligned} |\nabla_x (d_g(x, y) + d_g(x, y') - d_g(x, \tilde{y}) - d_g(x, \tilde{y}'))| &\geq c_0 \theta \\ \text{if } y \in \mathcal{T}_{C\theta}(\gamma_\mu), y' \in \mathcal{T}_{C\theta}(\gamma_{\mu'}), \tilde{y} \in \mathcal{T}_{C\theta}(\gamma_{\tilde{\mu}}), \text{ and } \tilde{y}' \in \mathcal{T}_{C\theta}(\gamma_{\tilde{\mu}'}), \end{aligned}$$

for some uniform $c_0 > 0$. Thus, (3.10) follows from Lemma 3.2 and a simple integration by parts argument since we are assuming that $\theta \geq \theta_0 = \lambda^{-\frac{1}{2} + \varepsilon_0}$ with $\varepsilon_0 > 0$.

4. Relationships with Zygmund's L^4 -toral eigenfunction bounds.

Recall that for \mathbb{T}^2 Zygmund [15] showed that if e_λ is an eigenfunction on \mathbb{T}^2 , i.e.,

$$(4.1) \quad e_\lambda(x) = \sum_{\{\varepsilon \in \mathbb{Z}^2: |\ell| = \lambda\}} a_\ell e^{ix \cdot \ell},$$

then

$$\|e_\lambda\|_{L^4(\mathbb{T}^2)} \leq C,$$

for some uniform constant C .

As observed in [2], using well-known pointwise estimates in two-dimensions, one has

$$\sup_{\gamma \in \Pi} \int_\gamma |e_\lambda|^2 ds = O_\varepsilon(\lambda^\varepsilon),$$

for all $\varepsilon > 0$. This of course implies that one also has

$$\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |e_\lambda|^2 dx = O_\varepsilon(\lambda^{-\frac{1}{2} + \varepsilon}),$$

for any $\varepsilon > 0$.

Sarnak [8] made an interesting observation that having $O(1)$ geodesic restriction bounds for \mathbb{T}^2 , is equivalent to the statement that there is a uniformly bounded number of lattice points on arcs of λS^1 of aperture $\lambda^{-\frac{1}{2}}$.¹

Using (1.1) we can essentially recover Zygmund's bound and obtain $\|e_\lambda\|_{L^4(\mathbb{T}^2)} = O_\varepsilon(\lambda^\varepsilon)$ for every $\varepsilon > 0$. (Of course this just follows from the pointwise estimate, but it shows how the method is natural too.)

If we could push the earlier results to include $\varepsilon_0 = 0$ and if we knew that there were uniformly bounded restriction bounds, then we would recover Zygmund's estimate.

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¹Cilleruelo and Córdoba [4] showed that this is the case for arcs of aperture $\lambda^{-\frac{1}{2}-\delta}$ for any $\delta > 0$.